CR compactification for asymptotically locally complex hyperbolic almost Hermitian manifolds

[arXiv:2307.04062]

Einstein Spaces and Special Geometry – Mittag-Leffler Institute

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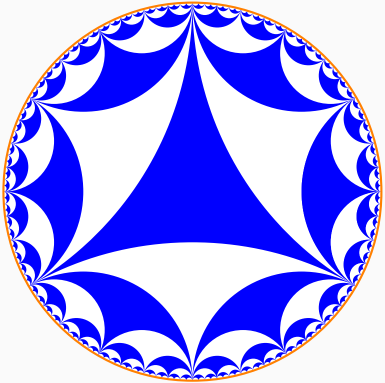
KTH Royal Institute of Technology
Stockholm



$(S^n, [\hat{g}])$ the conformal round sphere

$(\mathbb{H}^{n+1}, g)$ the hyperbolic space

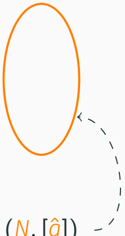
$(\overline{B^{n+1}(0;1)}, \overline{g})$ the unit ball

$\rho(x) = \frac{1-|x|^2}{2}$ is such that $\overline{g} = \rho^2 g$ restricts as $\hat{g}$ on S^n

$$\mathrm{Isom}(\mathbb{H}^{n+1}) = \mathrm{Conf}(S^n)$$

[Fefferman-Graham '85]

$(N, [\hat{g}])$ conformal manifold



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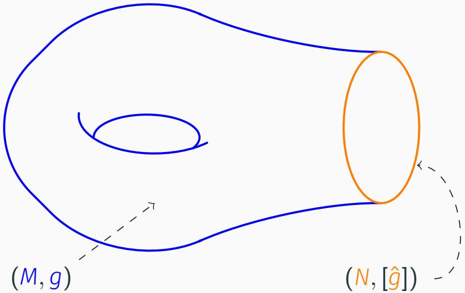

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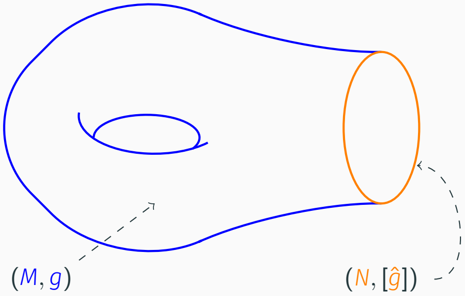

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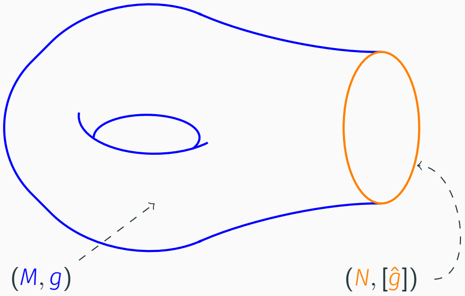

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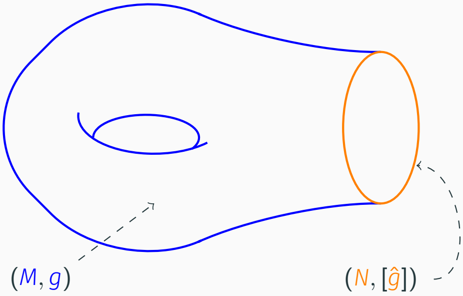

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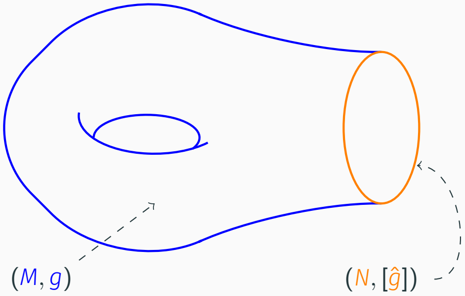
Riemannian invariants $\Longrightarrow$ conformal invariants

Sectional curvature



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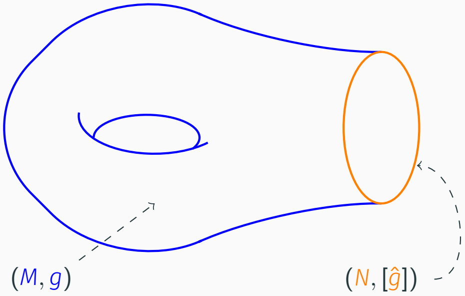

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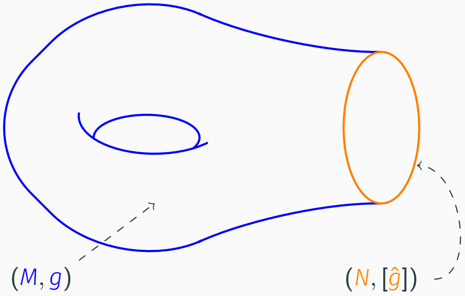
- **asymptotically hyperbolic** if $|d\rho|^2_{g} = 1$ on N
- **Poincaré-Einstein** if moreover $\mathrm{Ric}(g) = -ng$



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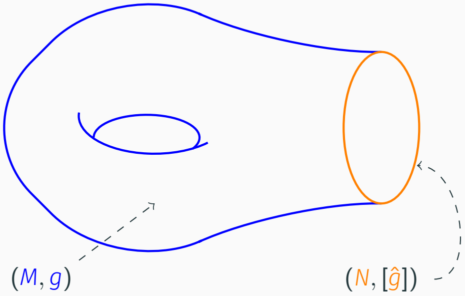
Graham-Lee '91, Biquard '00, Lee '06: perturbation of Poincaré-Einstein metrics

$\Longrightarrow$ new examples of Einstein metrics



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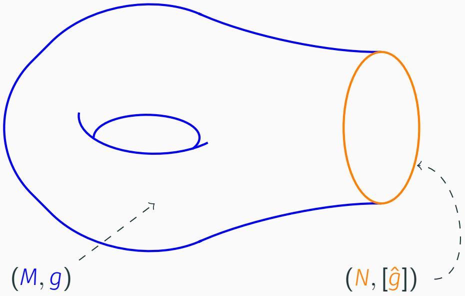

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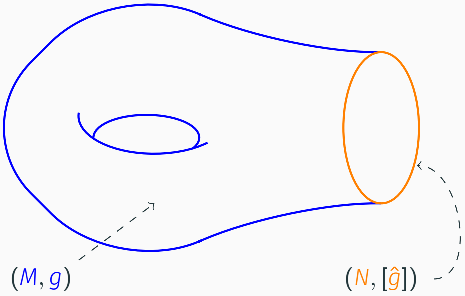

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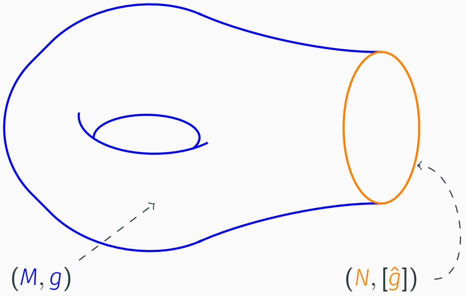

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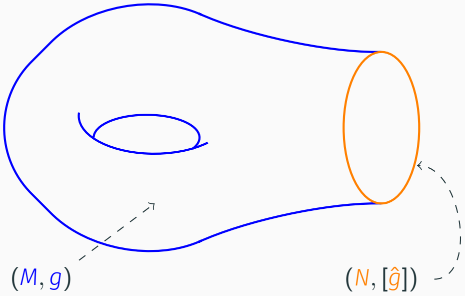

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$$\theta_p(v) = \frac{1}{2}\langle v, ip\rangle$$

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- H hyperplane distribution
- J_H formally integrable almost complex structure on H

Strictly pseudoconvex if

- H contact
- $\exists\theta$ calibrating H with $d\theta|_{\ker\theta}(\cdot, J_H\cdot) > 0$

$$\mathrm{Isom}(\mathbb{CH}^{n+1}) = \mathrm{Aut}_{CR}(S^{2n+1})$$

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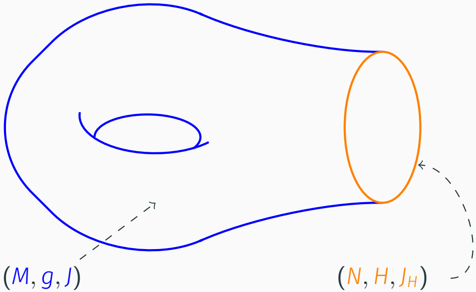
Intrinsic geometry of $(\mathbb{CH}^{n+1}, g, J) \implies$ CR geometry of S^{2n+1}



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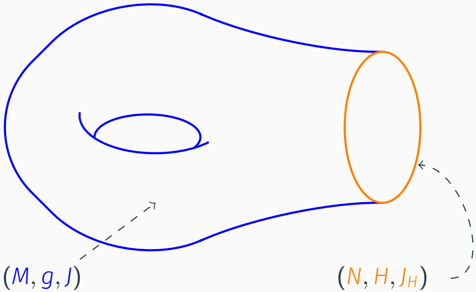

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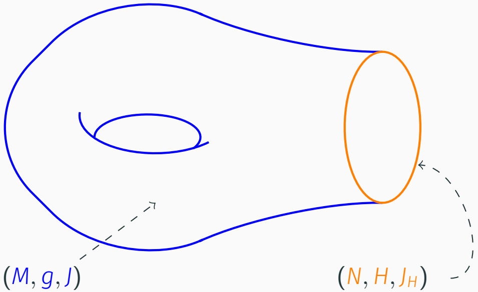

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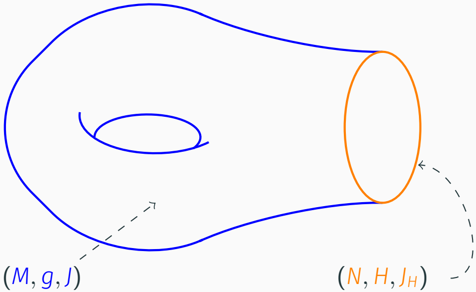
(almost) Hermitian / Kähler invariants $\Longrightarrow$ CR invariants



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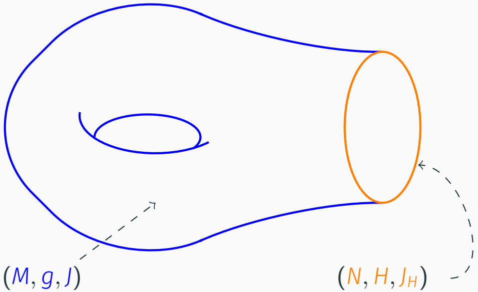

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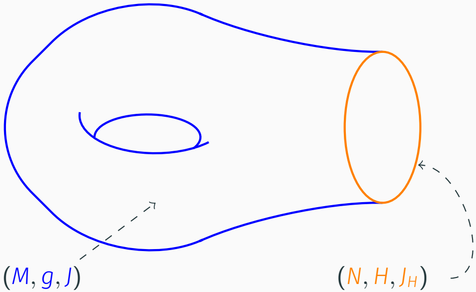

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perturbation of Kähler-Einstein metrics
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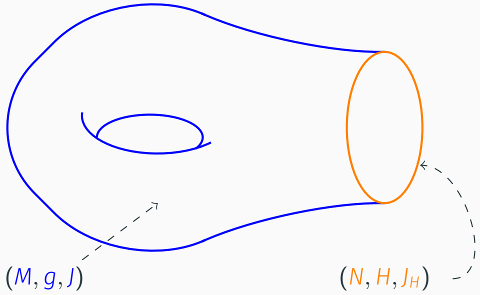

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Theorem [P. '22 –'23]

Let (M, g, J) be a complete, non-compact, almost Hermitian manifold of real dimension $\geqslant 4$, with a convex core. Assume that

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Moreover, g is asymptotically complex hyperbolic: there exist ρ defining function, η^0 a $\mathcal{C}^1$ contact form on $\partial\overline{M}$, such that

$$g = \frac{d\rho \otimes d\rho + \eta^0 \otimes \eta^0 + \rho\gamma + o(\rho^{3/2})}{\rho^2}$$

Comments

Bland 80's:

Compactification for some open Kähler manifolds

- assumptions not totally geometric
- $|R - R^0|_g = O(e^{-4r})$ *a posteriori*
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P. '21-'23

- $a > 3/2 \implies \mathcal{C}^1$ everywhere
- g Kähler $\implies$ assumptions on J are superfluous

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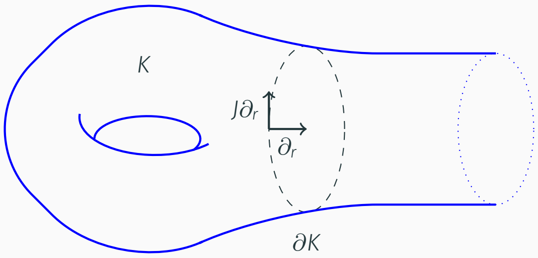


Convex core

- K compact, $\mathrm{codim} = 0$
- ∂K smooth, orientable
- $\sec(M \setminus K) < 0$

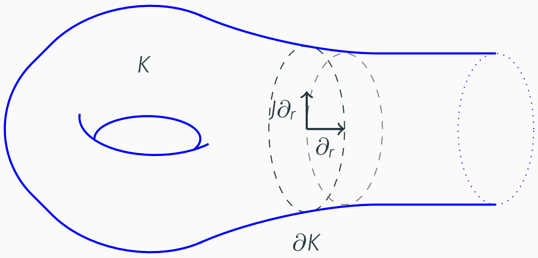
Normal exponential map

$$\mathcal{E}\colon (0,\infty) \times \partial K \xrightarrow{\sim} \overline{M \setminus K}$$

$$\mathcal{E}(r,p) = \mathcal{E}_r(p) := \exp_p(r\nu(p))$$

Intrinsic data

- Radial vector field ∂_r
- $J\partial_r$
- $\{\partial_r, J\partial_r\}^{\perp}$



Convex core

- K compact, $\mathrm{codim} = 0$
- ∂K smooth, orientable
- $\sec(M \setminus K) < 0$

Normal exponential map

$$\mathcal{E}\colon (0,\infty) \times \partial K \xrightarrow{\sim} \overline{M \setminus K}$$

$$\mathcal{E}(r,p) = \mathcal{E}_r(p) := \exp_p(r\nu(p))$$

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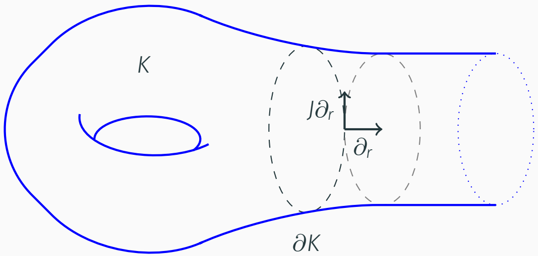

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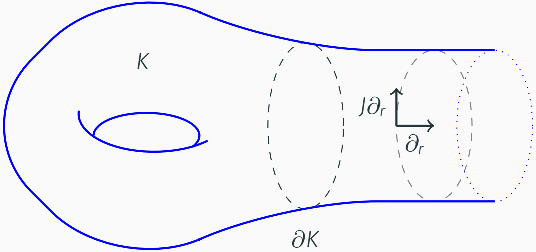

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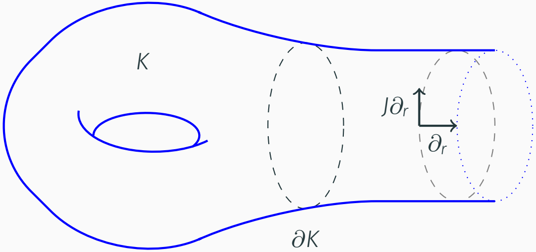

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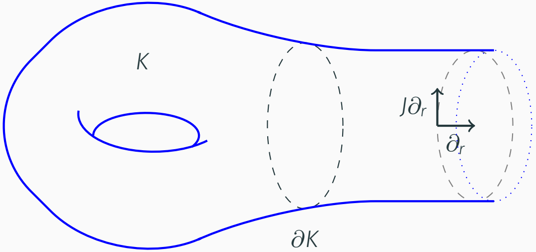

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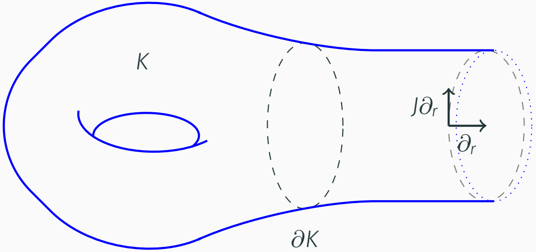

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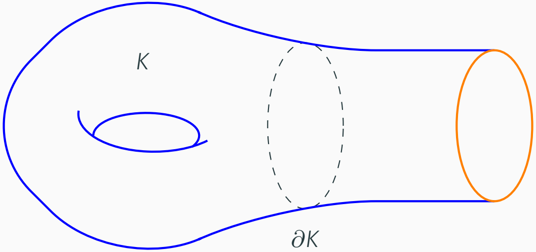

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Admissible frame: $\{\partial_r, J\partial_r, E_1, \ldots, E_{2n}\}$ orthonormal frame with $\nabla_{\partial_r} E_j = 0$

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$$g = dr \otimes dr + e^{2r}\eta_r^0 \otimes \eta_r^0 + e^r \gamma_r$$

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Relations

$$\begin{aligned}
{\varphi_r}^2 &= -\mathrm{Id} + \eta_r^0 \otimes \xi_0^r \\
\eta_r^0 \circ \varphi_r &= 0 \\
\gamma_r(\varphi_r \cdot, \varphi_r \cdot) &= \gamma_r \\
d\eta_r^0(\cdot, \varphi_r \cdot) &= \gamma_r + O(e^{(1-a)r})
\end{aligned}$$

with $\xi_0^r = \mathcal{E}_r^*(e^r J\partial_r)$

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Convergence of the coframes

$$\begin{cases} \partial^2 \eta_r^0 + 2\partial_r \eta_r^0 & = \sum_{k=0}^{2n} O\left(e^{(1/2-a)r}\right) \eta_r^k \\ \partial^2 \eta_r^j + \partial_r \eta_r^j & = \sum_{k=0}^{2n} O\left(e^{(1/2-a)r}\right) \eta_r^k \end{cases}$$

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There exists a unique vector field E_0 on $\overline{M \setminus K}$ such that

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Same strategy, but

- replace $J\partial_r$ by E_0 almost everywhere
- large amount of extra estimates
- extra error terms in all equations must be understood

Tack så mycket!