

## Quotient topology & some classic manifolds

### Correction

Let us recall a few definitions.

#### Definitions.

Let  $X$  be a topological space.

- $X$  is *compact* if any open cover has a finite subcover, that is:

$$\forall \{U_i\}_{i \in I} \text{ open subsets such that } X = \bigcup_{i \in I} U_i, \exists i_1, \dots, i_n \in I \text{ such that } X = \bigcup_{k=1}^n U_{i_k}.$$

- $Y \subset X$  is compact if it is compact for the induced topology.
- $X$  is *Hausdorff* if any two distinct points have disjoint neighbourhoods:

$$\forall x, y \in X, x \neq y, \exists U_x, U_y \text{ open such that } x \in U_x, y \in U_y \text{ and } U_x \cap U_y = \emptyset.$$

- $X$  is *locally compact* if for any point  $x \in X$  and any open neighbourhood  $V$  of  $x$ , there exists a smaller neighbourhood of  $x$  whose closure is compact and contained in  $U_x$  :

$$\forall x \in X, \forall V \text{ open neighb. of } x, \exists U \subset V \text{ open with } \bar{U} \text{ compact, } x \in U \subset \bar{U} \subset V.$$

If  $X$  is in addition Hausdorff, the following is equivalent: any point has a compact neighbourhood.

Let us also recall a few properties of topological spaces and continuous maps.

**Lemma 1.** *Let  $f: K \rightarrow Y$  be a continuous map with  $K$  compact. Then  $f(K)$  is compact.*

*Proof.* Suppose  $f(K) \subset \bigcup_{i \in I} U_i$  is an open cover. Then  $K \subset f^{-1}(f(K)) \subset \bigcup_{i \in I} f^{-1}(U_i)$  is an open cover (because  $f$  is continuous) of  $K$ , which is compact. Therefore, there exists  $i_1, \dots, i_n$  with  $K = \bigcup_{j=1}^n f^{-1}(U_{i_j})$ . It follows that  $f(K) \subset \bigcup_{j=1}^n U_{i_j}$  is a finite subcover, and that  $f(K)$  is compact.  $\square$

**Lemma 2.** *Let  $X$  be Hausdorff and  $K \subset X$  be compact. Then  $K$  is closed.*

*Proof.* Let us show that the complement  $X \setminus K$  is open. Let  $x \in X \setminus K$ . As  $X$  is Hausdorff, for any  $y \in K$ , one can find open neighbourhoods  $U_y \ni x$  and  $V_y \ni y$  such that  $U_y \cap V_y = \emptyset$ . As  $K \subset \bigcup_{y \in K} V_y$  and  $K$  is compact, there exists a finitely many points  $y_1, \dots, y_n$  with  $K \subset \bigcup_{j=1}^n V_{y_j}$ . It follows that  $U = \bigcap_{j=1}^n U_{y_j}$  is an open neighbourhood of  $x$  with  $U \subset X \setminus K$ , and  $X \setminus K$  is open.  $\square$

**Lemma 3.** *Suppose  $X$  is compact and  $F \subset X$  is closed, then  $F$  is compact.*

*Proof.* Suppose  $F \subset \bigcup_{i \in I} U_i$  is an open cover. Then  $X = F \cup (X \setminus F) = (\bigcup_{i \in I} U_i) \cup (X \setminus F)$  is an open cover of  $X$ . By compactness, there exists  $i_1, \dots, i_n$  such that we have a finite subcover  $X = (\bigcup_{i=1}^n U_{i_j}) \cup (X \setminus F)$ . It follows that  $F \subset \bigcup_{j=1}^n U_{i_j}$  is a finite subcover, and  $F$  is compact.  $\square$

**Exercise 1** (Quotient topology).

1.  $U \subset X/\sim$  is open  $\iff p^{-1}(U) \subset X$  is open.

2. Let  $f: X/\sim \longrightarrow Y$ . Then:

$$\begin{aligned} f \text{ is continuous} &\iff \forall U \subset Y \text{ open, } f^{-1}(U) \subset X/\sim \text{ is open} \\ &\iff \forall U \subset Y \text{ open, } p^{-1}(f^{-1}(U)) \subset X \text{ is open} \\ &\iff \forall U \subset Y \text{ open, } (f \circ p)^{-1}(U) \subset X \text{ is open} \\ &\iff f \circ p \text{ is continuous.} \end{aligned}$$

3. Consider  $\bar{x} \neq \bar{y} \in X/G$  and fix  $x \in p^{-1}(\{\bar{x}\})$ ,  $y \in p^{-1}(\{\bar{y}\})$ . Note that for all  $g \in X$ ,  $x \neq g \cdot y$ .

First,  $X$  is locally compact, so that there exists  $x \in U_x \subset K_x$  and  $y \in U_y \subset K_y$  with  $U_x$  and  $U_y$  open and  $K_x$  and  $K_y$  compact. Define  $K = K_x \cup K_y$ . As  $G \curvearrowright X$  is properly discontinuous, the set

$$\{g \in G \mid K \cap gK \neq \emptyset\}$$

is finite, and so is the set

$$\{g \in G \mid U_x \cap gU_y \neq \emptyset\}.$$

Call its elements  $g_1, \dots, g_n$ . The two  $G$ -invariant open sets

$$\bigcup_{g \in G} gU_x \text{ and } \bigcup_{g \in G} gU_y$$

may overlaps precisely because of the elements finitely many elements  $g_1, \dots, g_n$ : indeed, if  $g, h \in G$  are such that  $gU_x \cap hU_y \neq \emptyset$ , then  $U_x \cap (g^{-1}h)U_y \neq \emptyset$ . It follows that  $K \cap (g^{-1}h)K \neq \emptyset$  and thus,  $g^{-1}h \in \{g_1, \dots, g_n\}$ . Let us correct these overlaps.

The set  $X$  is Hausdorff, and for all  $j \in \{1, \dots, n\}$ ,  $x \neq g_j \cdot y$ ; thus, there exists  $V_j \ni x$  and  $W_j \ni g_j \cdot y$  open neighbourhoods with  $V_j \cap W_j = \emptyset$ . Define

$$V = U_x \cap \left( \bigcap_{j=1}^n V_j \right) \text{ and } W = U_y \cap \left( \bigcap_{j=1}^n g_j^{-1}W_j \right).$$

By construction,  $V$  and  $W$  are open neighbourhoods of  $x$  and  $y$  respectively such that

$$\forall g \in G, V \cap gW = \emptyset.$$

Denoting by  $\tilde{V} = \bigcup_{g \in G} gV$  and  $\tilde{W} = \bigcup_{g \in G} gW$ , they are disjoint  $G$ -invariant open subsets by construction, with  $x \in \tilde{V}$  and  $y \in \tilde{W}$ , so that  $p(\tilde{V}) \cap p(\tilde{W}) = \emptyset$ . But

$$p^{-1}(p(\tilde{V})) = \tilde{V}, \quad p^{-1}(p(\tilde{W})) = \tilde{W},$$

so that  $p(\tilde{V})$  and  $p(\tilde{W})$  are open in  $X/G$ . It follows that they are disjoint open neighbourhoods of  $\bar{x}$  and  $\bar{y}$ , and  $X/G$  is Hausdorff.

4. The action  $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$  defined by  $v \cdot x = x + v$  is properly discontinuous: indeed, if  $K \subset \mathbb{R}^n$  is compact with  $D = \text{diam}_{\|\cdot\|_\infty} K < +\infty$ , and if  $v \in \mathbb{Z}^n$  is such that  $\|v\|_\infty \geq D + 1$ , then  $K \cap (K + v) = \emptyset$ . It follows that

$$\{v \in \mathbb{Z}^n \mid K \cap (K + v) \neq \emptyset\} \subset \overline{B}(0, D + 1) \cap \mathbb{Z}^n$$

is a discrete subset of the compact  $\overline{B}(0, D + 1)$ : it is therefore finite. By 3.,  $\mathbb{T}^n$  is Hausdorff.

Note that the restriction  $p|_{[0,1]^n}: [0,1]^n \rightarrow \mathbb{T}^n$  is surjective and continuous. It follows that  $\mathbb{T}^n$  is compact as the continuous image of a compact.

Finally, if  $U$  is open in  $\mathbb{R}^n$ , then  $p^{-1}(p(U)) = \cup_{v \in \mathbb{Z}^n} (U + v)$  is open as a union of open subsets ( $U + v$  is homeomorphic to  $U$ ). Therefore  $p$  is an open map.

5. In order to show that  $f$  is a homeomorphism, let us show that  $g = f^{-1}$  is continuous. We will use the characterisation by closed subsets:  $g$  is continuous if and only if  $\forall F \subset K$  closed,  $g^{-1}(F) \subset K$  is closed.

Fix  $F \subset K$  a closed subset. Then  $F$  is compact as a closed subset of  $K$  compact. It follows that  $g^{-1}(F) = f(F)$  is compact as the continuous image of  $F$  compact by  $f$ . Now, recall that  $Y$  is Hausdorff, so that  $F \subset Y$  is closed. The result follows.

A counterexample is given by  $\text{id}: (X, \tau_1) \rightarrow (X, \tau_2)$ , with  $X = \{0, 1\}$ ,  $\tau_1 = \mathcal{P}(X)$  and  $\tau_2 = \{\emptyset, X\}$ .

6. The map  $f: t \in \mathbb{R} \mapsto e^{2i\pi t} \in \mathbb{S}^1$  is a surjective group homomorphism. Moreover, its kernel is  $\mathbb{Z}$ , which acts properly discontinuously on  $\mathbb{R}$ . It thus induces a continuous bijection

$$\overline{f}: \mathbb{T}^1 \rightarrow \mathbb{S}^1.$$

Recall that  $\mathbb{T}^1$  is compact Hausdorff by 4., and that  $\mathbb{S}^1$  is Hausdorff as a subspace of  $\mathbb{C}$ . Thus, it is a homeomorphism by 5.

More generally, the exact same study with  $f_n: \mathbb{R}^n \rightarrow (\mathbb{S}^1)^n$  defined by  $f_n(t_1, \dots, t_n) = (e^{2i\pi t_1}, \dots, e^{2i\pi t_n})$  yields a homeomorphism  $\overline{f}_n: \mathbb{T}^n \rightarrow (\mathbb{S}^1)^n$ .

7. Apparently, we do not have any properly discontinuous action of a discrete group. We have to find one.

Consider the inclusion map  $i: \mathbb{S}^n \rightarrow \mathbb{R}^{n+1} \setminus \{0\}$ . It is clearly continuous. Consider now the composition

$$q = p \circ i: \mathbb{S}^n \longrightarrow \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{RP}^n.$$

By 1.,  $q$  is continuous. Moreover, it is surjective as any (linear) line in  $\mathbb{R}^{n+1}$  intersects  $\mathbb{S}^n$ . It follows that  $\mathbb{RP}^n = q(\mathbb{S}^n)$ . The unit sphere being compact (for example, it is bounded and closed in the finite dimensional linear space  $\mathbb{R}^{n+1}$ ), so is  $\mathbb{RP}^n$ .

Define the antipodal action  $\{\pm 1\} \curvearrowright \mathbb{S}^n$  to be  $k \cdot x = kx$ . As  $\{\pm 1\}$  is finite, it is clearly a properly discontinuous action, and by 2., the quotient  $\mathbb{S}^n / \{\pm 1\}$  is Hausdorff. It is also compact, being the image of the quotient map  $\pi: \mathbb{S}^n \rightarrow \mathbb{S}^n / \{\pm 1\}$ .

Notice that if  $x \in \mathbb{S}^n$ , then  $q^{-1}(\{q(x)\}) = \{\pm x\}$ , so that  $q$  descends as a quotient map

$$\overline{q}: \mathbb{S}^n / \{\pm 1\} \longrightarrow \mathbb{RP}^n$$

such that  $q = \bar{q} \circ \pi$ . By 1.,  $\bar{q}$  is continuous. The map  $\bar{q}$  is hence a continuous bijection between  $\mathbb{S}^n / \{\pm 1\}$ , which is compact Hausdorff, and  $\mathbb{R}\mathbb{P}^n$ , which is compact. By 5., it is a homeomorphism, and finally,  $\mathbb{R}\mathbb{P}^n$  is compact Hausdorff.

Let us show that  $p$  is an open map. Let  $U \subset \mathbb{R}^{n+1} \setminus \{0\}$  be an open subset. Then

$$p^{-1}(p(U)) = \bigcup_{r \neq 0} rU$$

where  $rU = \{rx \mid x \in U\}$ . But  $rU$  is the preimage of  $U$  by the continuous map  $h_r: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}$  defined by  $h_r(x) = \frac{x}{r}$ . It follows that  $rU$  is open and therefore,  $p^{-1}(p(U))$  is open as a union of open subsets. Finally,  $p(U)$  is open, and  $p$  is an open map.

**Exercise 2** (The sphere).

1. The unit sphere is Hausdorff and second-countable as a subspace of  $\mathbb{R}^{n+1}$  which is both. Let  $N = (0, \dots, 0, 1) \in \mathbb{S}^n$  and  $S = (0, \dots, 0, -1) \in \mathbb{S}^n$  be the north and south pole of  $\mathbb{S}^n$  and  $U_N = \mathbb{S}^n \setminus \{N\}$ ,  $U_S = \mathbb{S}^n \setminus \{S\}$ .  $\{U_N, U_S\}$  is an open cover of  $\mathbb{S}^n$ . Consider the stereographic projections

$$p_N: \mathbb{S}^n \setminus \{N\} \longrightarrow \mathbb{R}^n, \quad p_S: \mathbb{S}^n \setminus \{S\} \longrightarrow \mathbb{R}^n$$

$$(X, t) \longmapsto \frac{X}{1-t}, \quad (X, t) \longmapsto \frac{X}{1+t}$$

where  $(X, t)$  refers to a point of  $\mathbb{S}^n$  seen as a subset of  $\mathbb{R}^n \times \mathbb{R}$ .

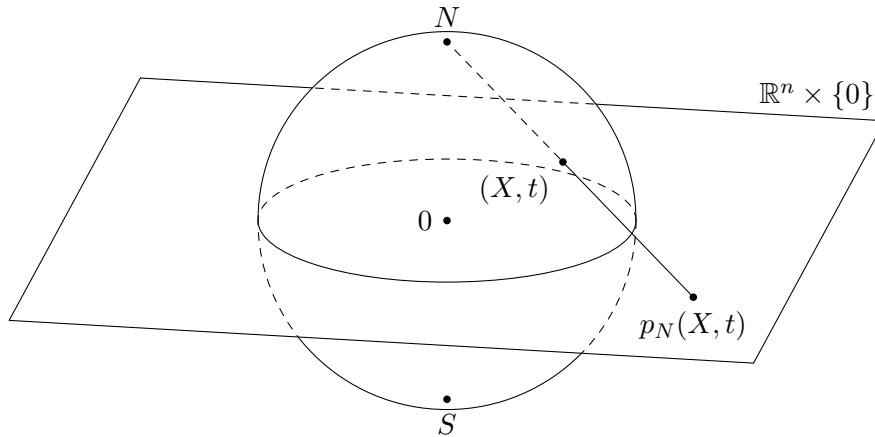


Figure 1: The stereographic projection from the north pole

They are homeomorphisms: indeed, one can check that they are continuous and that they have inverse

$$p_N^{-1}: \mathbb{R}^n \longrightarrow \mathbb{S}^n \setminus \{N\}, \quad p_S^{-1}: \mathbb{R}^n \longrightarrow \mathbb{S}^n \setminus \{S\}$$

$$x \longmapsto \left( \frac{2x}{\|x\|^2 + 1}, \frac{\|x\|^2 - 1}{\|x\|^2 + 1} \right), \quad x \longmapsto \left( \frac{2x}{\|x\|^2 + 1}, \frac{1 - \|x\|^2}{\|x\|^2 + 1} \right)$$

which are continuous.

Moreover, as  $U_N \cap U_S = \mathbb{S}^n \setminus \{N, S\}$ , it holds that  $p_N(U_N \cap U_S) = p_S(U_N \cap U_S) = \mathbb{R}^n \setminus \{0\}$ , and the transition map  $p_N \circ p_S^{-1}$  is given by

$$p_N \circ p_S^{-1}(x) = p_N \left( \frac{2x}{\|x\|^2 + 1}, \frac{1 - \|x\|^2}{\|x\|^2 + 1} \right) = \frac{\frac{2x}{\|x\|^2 + 1}}{1 - \frac{1 - \|x\|^2}{\|x\|^2 + 1}} = \frac{x}{\|x\|^2},$$

which is indeed smooth. Note that it is involutive so that  $p_S \circ p_N^{-1}$  is also smooth. It follows that  $\{(p_N, U_S), (p_S, U_S)\}$  is a smooth atlas. Considering the maximal atlas containing it endows  $\mathbb{S}^n$  with a smooth manifold structure.

## 2. Here is a slightly different proof than the one given orally.

Let us show that the two stereographic projections are restrictions of two ambient charts in  $\mathbb{R}^{n+1}$ . Consider the two geometric inversions

$$\begin{aligned} I_{N, \sqrt{2}}: \mathbb{R}^{n+1} \setminus \{N\} &\longrightarrow \mathbb{R}^{n+1} \setminus \{N\} & , & & I_{S, \sqrt{2}}: \mathbb{R}^{n+1} \setminus \{S\} &\longrightarrow \mathbb{R}^{n+1} \setminus \{S\} \\ x &\longmapsto N + 2 \frac{x - N}{\|x - N\|^2} & & & x &\longmapsto S + 2 \frac{x - S}{\|x - S\|^2} \end{aligned}$$

They are the geometric inversions in the spheres of radius  $\sqrt{2}$  and centers  $N$  and  $S$  respectively. They are two charts for  $\mathbb{R}^{n+1}$  and their restrictions to  $\mathbb{S}^n$  are  $p_N$  and  $p_S$ . It follows that the stereographic projections  $p_N$  and  $p_S$  are induced by ambient charts, and that the differentiable structure of  $\mathbb{S}^n$  induced from the ambient space  $\mathbb{R}^{n+1}$  is compatible with the atlas  $\{(U_N, p_N), (U_S, p_S)\}$ . They thus define the same differentiable structure on the sphere.

### Exercise 3 (Product manifolds).

First, a product of two Hausdorff (resp. second-countable) spaces is Hausdorff (resp. second-countable), so  $M \times N$  is Hausdorff (resp. second-countable). Let us find a smooth atlas on  $M \times N$ .

Let  $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$  (respectively  $\{(V_\beta, \psi_\beta)\}_{\beta \in B}$ ) be the maximal smooth atlas of  $M^m$  (respectively of  $N^n$ ). Let us show that  $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}_{(\alpha, \beta) \in A \times B}$  is a smooth atlas of  $M \times N$ .

First, for all  $(\alpha, \beta) \in A \times B$ ,  $U_\alpha \times V_\beta$  is open in  $M \times N$  by definition of the product topology, and moreover:

$$\begin{aligned} \bigcup_{(\alpha, \beta) \in A \times B} U_\alpha \times V_\beta &= \bigcup_{\alpha \in A} \bigcup_{\beta \in B} U_\alpha \times V_\beta \\ &= \bigcup_{\alpha \in A} \left( U_\alpha \times \left( \bigcup_{\beta \in B} V_\beta \right) \right) \\ &= \bigcup_{\alpha \in A} U_\alpha \times N \\ &= \left( \bigcup_{\alpha \in A} U_\alpha \right) \times N \\ &= M \times N \end{aligned}$$

so that it is indeed an open cover.

In addition, if  $(\alpha, \beta) \in A \times B$ , then

$$\begin{aligned} \varphi_\alpha \times \psi_\beta: U_\alpha \times V_\beta &\longrightarrow \varphi_\alpha(U_\alpha) \times \psi_\beta(V_\beta) \subset \mathbb{R}^m \times \mathbb{R}^n \\ (x, y) &\longmapsto (\varphi_\alpha(x), \psi_\beta(y)) \end{aligned}$$

is continuous by definition of the product topology, bijective with inverse  $\varphi_\alpha^{-1} \times \psi_\beta^{-1}$ , which is also continuous. It is thus a homeomorphism.

Finally, if  $(\alpha_1, \beta_1)$  and  $(\alpha_2, \beta_2)$  are elements of  $A \times B$ , then

$$(\varphi_{\alpha_1} \times \psi_{\beta_1}) \circ (\varphi_{\alpha_2}, \psi_{\beta_2})^{-1} = (\varphi_{\alpha_1} \circ \varphi_{\alpha_2}^{-1}) \times (\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})$$

is defined from  $\varphi_{\alpha_2}(U_{\alpha_1} \cap U_{\alpha_2}) \times \psi_{\beta_2}(V_{\beta_1} \cap V_{\beta_2})$  to  $\varphi_{\alpha_1}(U_{\alpha_1} \cap U_{\alpha_2}) \times \psi_{\beta_1}(V_{\beta_1} \cap V_{\beta_2})$ , is also smooth.

Hence,  $M^m \times N^n$  is a smooth manifold of dimension  $m + n$ .

**Exercise 4** (The torus).

Recall that we have shown in Exercise 1.4. that  $p: \mathbb{R}^n \rightarrow \mathbb{T}^n$  is an open map.

1. Let us first consider the 1 dimensional case. We have an open map

$$p: \mathbb{R} \rightarrow \mathbb{T}^1$$

so that if  $\tilde{U}_1 = (0, 1)$  and  $\tilde{U}_2 = (-\frac{1}{2}, \frac{1}{2})$ , then  $U_1 = p(\tilde{U}_1)$  and  $U_2 = p(\tilde{U}_2)$  are open in  $\mathbb{T}^1$ . Note that

$$U_1 \cup U_2 = p(\tilde{U}_1) \cup p(\tilde{U}_2) = p(\tilde{U}_1 \cup \tilde{U}_2) = p\left(\left(-\frac{1}{2}, 1\right)\right) = \mathbb{T}^1,$$

so that  $\{U_1, U_2\}$  is an open cover of  $\mathbb{T}^1$ . Consider the two maps :

$$\varphi_1: U_1 \longrightarrow \tilde{U}_1, \quad \varphi_2: U_2 \longrightarrow \tilde{U}_2$$

defined so that  $\varphi_j(\bar{x})$  is the only element of  $\tilde{U}_j \cap p^{-1}(\{\bar{x}\})$ . As  $p$  is an open map,  $\varphi_j$  are continuous. Being (local) right inverse to  $p$ , they are homeomorphisms. Let us show that  $\{(U_j, \varphi_j)\}_{j \in \{1, 2\}}$  is a smooth atlas.

We have

$$U_1 \cap U_2 = \mathbb{T}^1 \setminus \left\{ \bar{0}, \bar{\frac{1}{2}} \right\}, \quad \varphi_1(U_1 \cap U_2) = (0, 1) \setminus \left\{ \frac{1}{2} \right\}, \quad \varphi_2(U_1 \cap U_2) = \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\}.$$

Therefore, the transition functions are:

$$\begin{aligned} \varphi_1 \circ \varphi_2^{-1}: \quad \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\} &\longrightarrow (0, 1) \setminus \left\{ \frac{1}{2} \right\} \\ x &\longmapsto \begin{cases} x + 1 & \text{if } x < 0, \\ x & \text{if } x > 0, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \varphi_2 \circ \varphi_1^{-1}: (0, 1) \setminus \{\frac{1}{2}\} &\longrightarrow (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\} \\ x &\longmapsto \begin{cases} x & \text{if } x < \frac{1}{2}, \\ x - 1 & \text{if } x > \frac{1}{2}. \end{cases} \end{aligned}$$

These two transition functions are both smooth. We thus have constructed a smooth atlas on  $\mathbb{T}^1$ .

Now, for the  $n$ -dimensional case, consider the open subsets

$$\forall (i_1, \dots, i_n) \in \{1, 2\}^n, U_{i_1 \dots i_n} = U_{i_1} \times \dots \times U_{i_n} \text{ and } \tilde{U}_{i_1 \dots i_n} = \tilde{U}_{i_1} \times \dots \times \tilde{U}_{i_n},$$

where  $\tilde{U}_k$  and  $U_k$  are defined in the one dimensional case. Similarly to the one dimensional case, we show that they form an open cover of  $\mathbb{T}^n$ . Define the functions

$$\varphi_{i_1 \dots i_n}: \frac{U_{i_1 \dots i_n}}{(x_1, \dots, x_n)} \longrightarrow \begin{aligned} &\tilde{U}_{i_1 \dots i_n} \\ &\longmapsto (\varphi_{i_1}(\bar{x}_1), \dots, \varphi_{i_n}(\bar{x}_n)). \end{aligned}$$

Check that  $\{(U_{i_1 \dots i_n}, \varphi_{i_1 \dots i_n})\}_{(i_1, \dots, i_n) \in \{1, 2\}^n}$  is a smooth atlas on  $\mathbb{T}^n$ .

2. Check that the map  $\bar{f}: \mathbb{T}^n \rightarrow (\mathbb{S}^1)^n$  defined in Exercise 1.6 is smooth while seen in the constructed charts. To see that  $\bar{f}^{-1}$  is smooth, use the inverse function theorem (in charts).

**Exercise 5** (The projective space).

This Exercices has not been covered in class (at least for one group). Please try to solve it by yourself before checking the correction!

1. Recall from Exercise 1.7. that  $p: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}\mathbb{P}^n$  is open. Notice that the subset  $V_i = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\} \mid x_i \neq 0\}$  is open in  $\mathbb{R}^{n+1} \setminus \{0\}$ . Therefore,  $U_i = p(V_i)$  is open in  $\mathbb{R}\mathbb{P}^n$ . Define

$$\begin{aligned} \Phi_i: V_i &\longrightarrow \mathbb{R}^n \\ (x_0, \dots, x_n) &\longmapsto \frac{1}{x_i} (x_0, \dots, \hat{x}_i, \dots, x_n). \end{aligned}$$

It is clearly continuous and surjective. Moreover, we have

$$\forall t \neq 0, \forall X \in V_i, \Phi_i(tX) = \Phi_i(X).$$

In fact, we have  $\Phi_i(X) = \Phi_i(Y) \iff \exists t \neq 0, X = tY$  (check that in that case,  $t = \frac{x_i}{y_i}$ ). It thus induces a continuous bijective map

$$\begin{aligned} \varphi_i: U_i &\longrightarrow \mathbb{R}^n \\ [x_0 : \dots : x_n] &\longmapsto \frac{1}{x_i} (x_0, \dots, \hat{x}_i, \dots, x_n) \end{aligned}$$

Let us show that  $\varphi_i$  is a homeomorphism : to do so, let us find a continuous inverse to  $\varphi_i$ . Consider

$$\begin{aligned} \Psi_i: \mathbb{R}^n &\longrightarrow V_i \\ (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) &\longmapsto (x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n). \end{aligned}$$

Then  $\Psi_i$  is continuous. It follows that  $\psi_i = p \circ \Psi_i$  is continuous. But  $\psi_i \circ \varphi_i = \text{id}_{\mathbb{R}^n}$  and  $\varphi_i \circ \psi_i = \text{id}_{U_i}$ , so that  $\varphi_i: U_i \rightarrow \mathbb{R}^n$  is a homeomorphism.

2. First,  $\bigcup_{i=0}^n V_i = \mathbb{R}^{n+1} \setminus \{0\}$  so that  $p(\bigcup_{i=0}^n V_i) = \bigcup_{i=0}^n U_i = \mathbb{R}\mathbb{P}^n$ , and  $\{U_i\}_{i \in \{0, \dots, n\}}$  is an open cover of  $\mathbb{R}\mathbb{P}^n$ . Moreover, if  $i \neq j$ ,  $U_i \cap U_j = \{[x_0 : \dots : x_n] \mid x_i, x_j \neq 0\}$  and

$$\varphi_i \circ \varphi_j^{-1}(x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_n) = \left( \frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{j-1}}{x_i}, \frac{1}{x_i}, \frac{x_{j+1}}{x_i}, \dots, \frac{x_n}{x_i} \right)$$

is a rational function, and hence is smooth. It follows that  $\{(U_i, \varphi_i)\}_{i \in \{0, \dots, n\}}$  is a smooth atlas for  $\mathbb{R}\mathbb{P}^n$ .

We already know that  $\mathbb{R}\mathbb{P}^n$  is Hausdorff from Exercice 1.7.

Finally, it is second-countable as the image of  $\mathbb{R}^{n+1} \setminus \{0\}$ , which is second-countable, by  $p$ , which is continuous and open.

We thus have shown that  $\mathbb{R}\mathbb{P}^n$  is a smooth manifold.

3. Let us show that  $p$  is smooth, that is, for all  $i \in \{0, \dots, n\}$ , the map

$$p_i = \varphi_i \circ p: p^{-1}(U_i) = \mathbb{R}^{n+1} \setminus \{x_i = 0\} \rightarrow \varphi_i(U_i) = \mathbb{R}^n$$

is smooth. Its expression is given by

$$p_i(x_0, \dots, x_n) = \frac{1}{x_i}(x_0, \dots, \widehat{x_i}, \dots, x_n)$$

which is indeed smooth as a rational function.

4. Let us show that  $p$  is a local diffeomorphism near the north pole  $N$ . Let  $D_N$  be the open upper hemisphere of  $\mathbb{S}^n$ , that is

$$D_N = \{(x_0, \dots, x_n) \in \mathbb{S}^n \mid x_n > 0\}$$

and consider the stereographic projection from the south pole  $p_S$  (see exercise 2.). Then  $p_S(D_N) = B_0(1) \subset \mathbb{R}^n$ . Thus,  $p_S|_{D_N}$  takes value in  $U_n$  and we can consider the composition

$$f = \varphi_n \circ p \circ p_S^{-1}: B_0(1) \rightarrow \mathbb{R}^n$$

which has the expression, for  $x = (x_1, \dots, x_n) \in B_0(1)$  :

$$f(x) = \varphi_n \circ p \left( \frac{2x}{\|x\|^2 + 1}, \frac{1 - \|x\|^2}{\|x\|^2 + 1} \right) = \varphi_n \left( \left[ \frac{2x}{\|x\|^2 + 1} : \frac{1 - \|x\|^2}{\|x\|^2 + 1} \right] \right) = \frac{2x}{1 - \|x\|^2}.$$

Let us compute its differential at 0. We have, for  $h$  small enough:

$$f(h) = \frac{2h}{1 - \|h\|^2} = 2h(1 + o(h)) = f(0) + 2h + o(h)$$

So that  $df(0) = 2 \text{id}$  is invertible. The inverse function Theorem then shows that  $f$  is a local diffeomorphism near 0. This means that  $p$  is a local diffeomorphism near the north pole  $N$ .

The exact same study near any point  $P \in \mathbb{S}^n$ , considering the stereographic projection from  $-P$ , shows that the projection  $p$  is a local diffeomorphism near  $P$ , and thus,  $p$  is a local diffeomorphism near any point. It is then a local diffeomorphism.



5. We have the stereographic atlas for  $\mathbb{S}^1$  given by  $\{(p_N, \mathbb{S}^1 \setminus \{N\}), (p_S, \mathbb{S}^1 \setminus \{S\})\}$ , with transition function

$$p_N \circ p_S^{-1}: \mathbb{R}^* \longrightarrow \mathbb{R}^* \\ t \longmapsto \frac{1}{t}$$

and the affine atlas for  $\mathbb{RP}^1$  given by  $\{(U_0, \varphi_0), (U_1, \varphi_1)\}$ , with transition function

$$\varphi_0 \circ \varphi_1^{-1}: \mathbb{R}^* \longrightarrow \mathbb{R}^* \\ t \longmapsto \frac{1}{t}.$$

Heuristically, these two manifolds are given by the same construction : they are two disjoint copies of  $\mathbb{R}$  where we identify the two disjoint  $\mathbb{R}^*$  with the inverse map. This allows us to construct a diffeomorphism. Define

$$f: \mathbb{RP}^1 \longrightarrow \mathbb{S}^1 \\ [x : y] \longmapsto \begin{cases} p_N^{-1} \circ \varphi_0([x : y]) & \text{if } [x : y] \in U_0 \\ p_S^{-1} \circ \varphi_1([x : y]) & \text{if } [x : y] \in U_1 \end{cases}$$

Check that  $f$  is well-defined (that is, its two expressions on  $U_0 \cap U_1$  give the same point in  $\mathbb{S}^1$ ) and that  $f$  is a diffeomorphism.