Submanifolds, Tangent spaces and Differentials
Critical values, Sard's Theorem
Correction

Exercise 1 (Tangent space of a submanifold)**.**

Let $M \subset \mathbb{R}^m$ and $N \in \mathbb{R}^n$ be submanifolds of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

1. (a) If M is locally given by a submersion $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^{n-m}$, with $M \cap U = f^{-1}(\{0\})$, then
$$T_pM = \ker\left(\mathrm{d}f(p)\right)$$
   (b) If M is locally given by an immersion $f\colon V \subset \mathbb{R}^m \to U \subset \mathbb{R}^n$, with $f(0) = p$ and f homeomorphism from V to $U \cap M$, then
$$T_pM = \mathrm{Im}\left(\mathrm{d}f(0)\right)$$
   (c) If M is locally given by a diffeomorphism $f\colon U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$, with $f(p) = 0$ and $M \cap U = f^{-1}\left(V \cap (\mathbb{R}^m \times \{0\})\right)$, then
$$T_pM = \mathrm{d}f^{-1}(0) \cdot (\mathbb{R}^m \times \{0\})$$

   Note that if M is considered as an abstract manifold in itself, and T_pM as an abstract vector space, then the inclusion map
$$\iota\colon M \to \mathbb{R}^n$$
   is a smooth embedding and its differential at p
$$\mathrm{d}\iota(p)\colon T_pM \to \mathbb{R}^n$$
   is an injective linear map with image $\mathrm{d}\iota(p)\left(T_pM\right)$ canonically isomorphic to the extrinsic definitions of T_pM above.

2. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open neighborhoods. Let $\widetilde{f} : U \to V$ be a smooth map such that $f := \widetilde{f}|_{M\cap U} : M \cap U \to N \cap V$. Show that f is a smooth map between manifolds and that
$$\mathrm{d}f_p = \left(\mathrm{d}\widetilde{f}_p\right)\Big|_{T_pM} : T_pM \to T_{f(p)}N.$$

Exercise 2 (Veronese embedding)**.**

1. It suffices to show that f is a homeomorphism onto its image. It will then be a smooth immersion which is an homeomorphism onto its image, that is, an embedding. Notice that it is sufficient to show that f^{-1} is continuous. Here are two proofs:

   (a) Using that a manifold is metrizable, *i.e* that its topology is defined thanks to a metric.

   Let $\{y_n\}_{n\in\mathbb{N}}$ be a converging sequence in $f(M)$ with limit $y \in f(M)$. Let x_n and x be the unique point in M with $f(x_n) = y_n$ and $f(x) = y$ (which do exist because f is injective). Then the subset $K = (\cup_{n\in\mathbb{N}}\{y_n\}) \cup \{y\}$ is compact. The function f

being proper, $f^{-1}(K) = (\cup_{n\in\mathbb{N}}\{x_n\}) \cup \{x\}$ is compact. Hence, $\{x_n\}_{n\in\mathbb{N}}$ has a limit points. Let $\widetilde{x}$ be on of them. By continuity of f, we have $f(x_n) \to f(\widetilde{x})$, that is $y_n \to f(\widetilde{x})$, and $f(x) = f(\widetilde{x})$. By injectivity of f, $\widetilde{x} = x$. It follows that (x_n) has a unique limit point x, and thus, $x_n \to x$, that is, $f^{-1}(y_n) \to f^{-1}(y)$. Finally, f^{-1} is continuous.

(b) Using that a manifold is compactly generated, that is a subset is open / closed if and only if it is open / closed in any compact subset. In particular, it suffices to show that the restriction of f^{-1} to any compact subset of $f(M)$ is continuous.

Let $K \subset f(M)$ be a compact: f being proper, $f^{-1}(K)$ is compact. Then, the restriction $f|_{f^{-1}(K)} \colon f^{-1}(K) \to K$ is continuous between $f^{-1}(K)$, compact Hausdorff, and K, compact. Hence, it is a homeomorphism. It follows that $f^{-1}|_K$ is continuous. This concludes the proof.

2. Let us first show that f is well defined. Let $(x:y:z) \in \mathbb{RP}^2$ be represented by $(x,y,z) \neq (0,0,0)$ and $(x',y',z') \neq (0,0,0)$: there exists $\lambda \neq 0$ such that $(x,y,z) = (\lambda x, \lambda y, \lambda z)$. But then, $(x^2, y^2, z^2, xy, yz, zx) = \lambda^2(x'^2, y'^2, z'^2, x'y', y'z', z'x') \neq (0,0,0,0,0,0)$ and it follows that f is well defined.

Let us show that f is injective. Suppose $f(x:y:z) = f(x':y':z')$. Then there exists $t \neq 0$ with

$$\begin{cases} x^2 &= tx'^2 \\ y^2 &= ty'^2 \\ z^2 &= tz'^2 \\ xy &= tx'y' \\ yz &= ty'z' \\ yx &= tz'x' \end{cases}$$

Notice first that $t > 0$. The three first equations shows that

$$\begin{cases} x &= \pm\sqrt{t}x' \\ y &= \pm\sqrt{t}y' \\ z' &= \pm\sqrt{t}z' \end{cases}$$

The fourth equation shows that either $(x,y) = \sqrt{t}(x',y')$ or $(x,y) = -\sqrt{t}(x',y')$: that is, the sign is the same for the two first equations. Similarly, the fifth equation shows that the sign before $\sqrt{t}$ is the same for y and z. It follows that $(x:y:z) = (x':y':z')$, and then, f is injective.

Let us now show that f is an immersion. Consider the affine chart (φ_x, U_x) of $\mathbb{RP}^2$ defined by $U_x = \{(x:y:z) \in \mathbb{RP}^2 \mid x \neq 0\}$ and $\varphi_x^{-1} \colon \mathbb{R}^2 \to U_x$ defined by $\varphi_x^{-1}(y,z) = (1:y:z)$. Similarly, consider the chart (U_0, φ_0) of $\mathbb{RP}^5$ defined by $U_a = \left\{(a:b:c:d:e:f) \in \mathbb{RP}^5 \mid a \neq 0\right\}$ and $\varphi_a(a:b:c:d:e:f) = \frac{1}{a}(b,c,d,e,f)$. Then $f(U_x) \subset U_a$ and in these charts, we have

$$\widetilde{f} = \varphi_a \circ f \circ \varphi_x^{-1} \colon \quad \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^5 \\ (z,y) & \longmapsto & (y^2, z^2, y, yz, z) \end{array}$$

It is smooth as a polynomial function, and if $(y,z) \in \mathbb{R}^2$, the differential of $\widetilde{f}$ at (x,y)

has matrix in the canonical bases

$$\mathrm{Mat}_{\mathrm{can}}\left(\mathrm{d}\widetilde{f}(x,y)\right) = \begin{bmatrix} 2y & 0 \\ 0 & 2z \\ 1 & 0 \\ z & y \\ 0 & 1 \end{bmatrix}$$

and thus, $\mathrm{d}\widetilde{f}(y,z)$ has rank 2. Hence, f is an immersion on U_x. Similarly, f is a smooth immersion on $U_y = \{(x:y:z) \in \mathbb{RP}^2 \mid y \neq 0\}$ and on $U_z = \{(x:y:z) \in \mathbb{RP}^2 \mid z \neq 0\}$ and finally, f is a smooth immersion on $U_x \cup U_y \cup U_z = \mathbb{RP}^2$.

Therefore, f is an injective smooth inmmersion on $\mathbb{RP}^2$. As $\mathbb{RP}^2$ is compact and f continuous, f is obviously proper. By 1., f is an embedding.

Exercise 3 (Tangent space of the torus)**.**

1. Consider the map $f\colon \mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$f(t_1,t_2) = (\cos 2\pi t_1, \sin 2\pi t_1, \cos 2\pi t_2, \sin 2\pi t_2)$$

It is $\mathbb{Z}^2$-invariant and descends as a smooth map $\overline{f}\colon \mathbb{T}^2 \to \mathbb{R}^4$ such that $f = \overline{f} \circ p$. If $(t_1,t_2) \in \mathbb{R}^2$, then $\mathrm{d}f(t_1,t_2)$ has matrix in the canonical bases

$$\mathrm{Mat}_{\mathrm{can}}\left(\mathrm{d}f(t_1,t_2)\right) = 2\pi \begin{bmatrix} -\sin 2\pi t_1 & 0 \\ \cos 2\pi t_1 & 0 \\ 0 & -\sin 2\pi t_2 \\ 0 & \cos 2\pi t_2 \end{bmatrix}$$

and $\mathrm{d}f(t_1,t_2)$ is of rank 2. Recall that $p\colon \mathbb{R}^2 \to \mathbb{T}^2$ is a local diffeomorphism. Therefore, if $(t_1,t_2) \in \mathbb{R}^2$, the chain rule yields

$$\mathrm{d}f(t_1,t_2) = \mathrm{d}\overline{f}(\overline{t_1,t_2}) \circ \mathrm{d}p(t_1,t_2)$$

from which we deduce, recalling that $\mathrm{d}p(t_1,t_2)$ is a linear isomorphism

$$\mathrm{d}\overline{f}(\overline{t_1,t_2}) = \mathrm{d}f(t_1,t_2) \circ \left(\mathrm{d}p(t_1,t_2)\right)^{-1}$$

Thus, $\mathrm{d}\overline{f}(\overline{t_1,t_2})$ has rank 2. This being true for all $\overline{(t_1,t_2)} \in \mathbb{T}^2$, $\overline{f}$ is an immersion.

One can check that $\overline{f}$ is injective: for example, $f\colon \mathbb{R}^2 \to \mathbb{C}^* \times \mathbb{C}^*$ is a group homomorphism with kernel $\mathbb{Z}^2$, so that $\overline{f}$ is injective.

Finally, $\mathbb{T}^2$ is compact Hausdorff, so that $\overline{f}$ is proper (any continuous function on a compact Haussdorff space is proper). Hence, $\overline{f}$ is an injective proper immersion, and is an embedding.

Let us now consider $\mathbf{T}^2 = \overline{f}(\mathbb{T}^2) \subset \mathbb{R}^4$ the embedded torus in $\mathbb{R}^4$ and fix any point $p = (\cos 2\pi t_1, \sin 2\pi t_1, \cos 2\pi t_2, \sin 2\pi t_2) \in \mathbf{T}_2$. Then $\mathbf{T}^2$ is locally given by the immersion $f\colon \mathbb{R}^2 \to \mathbb{R}^4$ defined above, and $T_p\mathbf{T}^2 = \mathrm{Im}\,\mathrm{d}f(t_1,t_2)$, which is then

$$T_p\mathbf{T}_2 = \left\{\lambda X_1(p) + \mu X_2(p) \mid (\lambda,\mu) \in \mathbb{R}^2\right\}$$

with $X_1(p) = \begin{bmatrix} -\sin 2\pi t_1 \\ \cos 2\pi t_1 \\ 0 \\ 0 \end{bmatrix}$ and $X_2(p) = \begin{bmatrix} 0 \\ 0 \\ -\sin 2\pi t_2 \\ \cos 2\pi t_2 \end{bmatrix}$, which are linearly independant.
It follows that X_1 and X_2 are smooth vector fields on $\mathbf{T}_2$ which are everuwhere linearly independant, so that $\mathbf{T}_2$ is trivializable.

2. If $R > r$, the map

$$g_{R,r}\colon \quad \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(t_1, t_2) \longmapsto \begin{pmatrix} (R + r\cos 2\pi t_1)\cos 2\pi t_2 \\ (R + r\cos 2\pi t_1)\sin 2\pi t_2 \\ r\sin 2\pi t_1 \end{pmatrix}$$

is $\mathbb{Z}^2$ invariant and descends to the quotient as $\overline{g}_{R,r}\colon \mathbb{T}^2 \to \mathbb{R}^3$ which is an embedding: this can be derived from the exact same study as in 1.

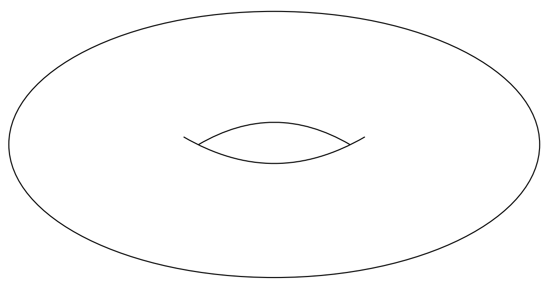

Figure 1: A torus embedded in $\mathbb{R}^3$

3. An exact same study as in 1. shows that one can embed $\mathbb{T}^n$ in $\mathbb{R}^{2n}$ and that the vector fields

$$X_i \begin{pmatrix} \cos 2\pi t_1 \\ \cos 2\pi t_2 \\ \vdots \\ \cos 2\pi t_n \\ \sin 2\pi t_n \end{pmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\sin 2\pi t_i \\ \cos 2\pi t_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

give a parallelization of $\mathbb{T}^n \subset \mathbb{R}^{2n}$.

Remark: we have shown that these particular embedded tori in $\mathbb{R}^4$ or $\mathbb{R}^3$ are parallelizable. In fact, this notion is intrinsic and does not depends on the embedding in $\mathbb{R}^N$, but it requires the definition of vector bundle isomorphism, which has not been seen yet.

Exercise 4 (Tangent space of spheres)**.**

1. Let $\gamma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ be a smooth arc. Then for all t, $\|\gamma(t)\|^2 = 1$. Differentiating this shows that $\langle \gamma(0), \gamma'(0) \rangle = 0$, so that $\gamma'(0) \in \gamma(0)^\perp$. We deduce that $T_{\gamma(0)}\mathbb{S}^1 \subset \gamma(0)^\perp$,

and we have in fact an equality because $\mathbb{S}^1$ is a one dimensional submanifold of $\mathbb{R}^2$, so that $T_{\gamma(0)}\mathbb{S}^1$ is one dimensional.

If follows that $X(x,y) = (-y,x)$ is a smooth vector field on $\mathbb{S}^1$, which does not vanish. Thus, we have found a parallelization of $T\mathbb{S}^1$.

The two dimensional sphere is not parallelizable because of the Hairy ball Theorem: any smooth (in fact, continuous) vector field on $\mathbb{S}^2$ vanishes somewhere.

2. Let $G \subset \mathbb{R}^n$ be a Lie group.

   (a) For $g \in G$, consider the left translation by g:

$$L_g\colon \begin{array}{ccc} G & \longrightarrow & G \\ h & \longmapsto & gh \end{array}$$

   which is smooth as the restiction to $\{g\} \times G$ of the multiplication μ. Then $L_{g^{-1}}$ is also smooth and $L_q \circ L_{g^{-1}} = L_{g^{-1}} \circ L_g = \mathrm{Id}_G$, so that L_g is a diffeomorphism.
   As $L_g(e) = g$, the linear map

$$\mathrm{d}L_g(e)\colon T_eG \to T_gG \subset \mathbb{R}^n$$

   is a linear isomorphism. It follows from the smoothness of μ that the map $g \in G \mapsto \mathrm{d}L_g(e) \in \mathcal{L}(T_eG, \mathbb{R}^n)$ is smooth.
   Let $(e_1, \dots e_m)$ be a basis of T_eG and consider $X_i\colon G \to \mathbb{R}^n$ defined by

$$X_i(g) = \mathrm{d}L_g(e) \cdot e_i$$

   By construction, it is a smooth vector field on G, and at each point $g \in G$, $(X_1(g), \dots, X_m(g))$ is a basis of T_gG. Thus, G is parallelizable.

   (b) It is a well-known fact that $SU(2)$ is a subgroup of $GL_2(\mathbb{C})$ with

$$SU(2) = \left\{ \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

   Consider the map

$$f\colon \begin{array}{ccc} \mathbb{R}^4 & \longrightarrow & M_2(\mathbb{C}) \\ (x,y,z,t) & \longmapsto & \begin{bmatrix} x+iy & z+it \\ -z+it & x-iy \end{bmatrix} \end{array}$$

   Then f is an injective linear map, and is thus a smooth embedding. Moreover, $\mathbb{S}^3$ is a submanifold of $\mathbb{R}^4$ whose image is precisely $SU(2)$: it follows that $SU(2)$ is a submanifold of $M_2(\mathbb{C}) \simeq \mathbb{R}^8$ diffeomorphic to $\mathbb{S}^3$.
   The multiplication in $M_2(\mathbb{C})$ is smooth, and so is its restriction to $SU(2)$. Also, the inversion in $GL_2(\mathbb{C})$ is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

   which is smooth, and its restriction to $SU(2)$ is smooth.

(c) We know that $SU(2)$ is parallelizable as a Lie group. Hence, it has 3 vectors fields Y_1, Y_2 and Y_3 that are pointwise linearly independant. It follows that $X_i(p) = \mathrm{d}f(p)^{-1}Y_i(f(p))$ are three vector fields on $\mathbb{S}^3$ which are pointwise linearly independant, and then, $\mathbb{S}^3$ is parallelizable.

(d) *(bonus)* In $\mathbb{C}^2$, define

$$\mathbf{1} = (1, 0), \qquad \mathbf{i} = (i, 0), \qquad \mathbf{j} = (0, 1), \qquad \mathbf{k} = (0, i),$$

which form an orthonormal basis if $\mathbb{C}^2 \simeq \mathbb{R}^4$ for the usual inner product. One can show that the multiplication defined by

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}$$

extended linearly, is associative (in fact, we have constructed the *quaternions*) and satisfies $\|qq'\| = \|q\|\|q'\|$.
Define on $\mathbb{S}^3$ the functions

$$X_1(x) = \mathbf{i}x, \qquad X_2(x) = \mathbf{j}x, \qquad X_3(x) = \mathbf{k}x.$$

Show that they are vector fields on $\mathbb{S}^3$ and that they parallelize the sphere.

Exercise 5 (Computation of a differential)**.**

Exercise 6 (Extending smooth function)**.**

Sketch of a proof:

1. First, note that if $f\colon M^m \subset \mathbb{R}^p \to N^n \subset \mathbb{R}^q$ is smooth, then its co-extension $f\colon M \to \mathbb{R}^q$ is smooth, and "there is nothing to tell on the right". Let us then focus on the left.

2. Second, take a chart on $\mathbb{R}^p$ adapted to M, that is, in that chart, M is given by $M \cap U = \{(x^1, \dots, x^m, 0, \dots, 0)\}$.

3. In that chart, extend f by $\widetilde{f}(x^1, \dots, x^p) = f(x^1, \dots, x^m)$.

4. Choose a locally finite open covering of M by charts as above and consider a partition of unity subordinate to this cover. Glue the extensions constucted above thanks to this partition of unity: this gives an extension of f on an open subset of $\mathbb{R}^p$.

5. Enjoy.

Exercise 7 (Critical points VS critical values)**.**

1. $\mathbb{R} \setminus F$ is an open subset of the real line. If non-empty, it is a countable union of disjoint open intervals: say $\mathbb{R} \setminus F = \cup_{i \in I}(a_i, b_i)$ with I finite or countable. It may be possible that one of the a_i (and only one) is equal to $-\infty$, and similarly, that one of the b_i (and only one) if equal to $+\infty$.

   For $i \in I$, construct a smooth nonnegative function $f_i\colon \mathbb{R} \to \mathbb{R}$ with $f_i(x) > 0 \iff x \in (a_i, b_i)$. Then the function $f = \sum_{i \in I} f_i$ is a solution.

2. As K is closed, there exists by 1. a smooth nonnegative function $g\colon \mathbb{R} \to \mathbb{R}$ with $g(x) = 0 \iff x \in K$. Define
$$f(x) = \int_0^x g(t)\,\mathrm{d}t.$$
Then f is smooth with $f' = g \geqslant 0$, and is thus nondecreasing. The function g vanishes exactly on K, which is of empty interior: hence, if $x < y$, then $f(y) - f(x) = \int_x^y g(t)\,\mathrm{d}t > 0$, and f is strictly increasing. If follows that f is a smooth homeomorphism of $\mathbb{R}$ onto its image. Its set of critical points is K, by definition of g, which has Lebesgue measure $\lambda(K) > 0$. But by Sard's Theorem, $f\left(\mathrm{Crit}(f)\right)$ has measure zero.

Remark: the function f constructed above may not be a homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, because its image may not be all of $\mathbb{R}$. But we can adapt the proof in order to do so: show it!

Exercise 8 (Change of variable)**.**

First, notice that we have the disjoint union $\varphi(U) = (\varphi(U) \setminus \varphi(\mathrm{Crit}\varphi)) \cup \varphi(\mathrm{Crit}\varphi)$. Therefore:

$$\lambda(\varphi(U)) = \int_{\varphi(U)} 1\,\mathrm{d}\lambda = \int_{\varphi(U)\setminus\varphi(\mathrm{Crit}\varphi)} 1\,\mathrm{d}\lambda + \int_{\varphi(\mathrm{Crit}\varphi)} 1\,\mathrm{d}\lambda$$

From Sard's Theorem, $\varphi(\mathrm{Crit}\varphi)$ has measure zero, and thus, we have

$$\lambda(\varphi(U)) = \int_{\varphi(U)\setminus\varphi(\mathrm{Crit}\varphi)} 1\,\mathrm{d}\lambda$$

Since φ is an homeomorphism, it is injective and it follows that $\varphi(U)\setminus\varphi(\mathrm{Crit}\varphi) = \varphi\left(U \setminus \mathrm{Crit}\varphi\right)$. But by definition of $\mathrm{Crit}\varphi$, $\mathrm{d}\varphi(x)$ is invertible is $x \in U \setminus \mathrm{Crit}\varphi$, and from the inverse function Theorem, the restriction $\varphi|_{U\setminus\mathrm{Crit}\varphi}$ is a diffeomorphism onto its image. The usual change of variable gives
$$\lambda(\varphi(U)) = \int_{U\setminus\mathrm{Crit}\varphi} |\det \mathrm{d}\varphi|\,\mathrm{d}\lambda$$
and to conclude, note that on $\mathrm{Crit}\varphi$, we have $|\det \mathrm{d}\varphi| = 0$, so that we have the formula

$$\lambda(\varphi(U)) = \int_U |\det \mathrm{d}\varphi|\,\mathrm{d}\lambda.$$