Differential forms, orientability, integration on manifolds

Exercise 1.

1. Let $n = \dim M$ and consider $X_1, \dots, X_n$ n everywhere linearly independant vector fields on M. Define the differential form of degree n on M

$$\forall p \in M, \quad \omega_p \colon \left| \begin{array}{ccc} (T_pM)^n & \longrightarrow & \mathbb{R} \\ (v_1, \dots, v_n) & \longmapsto & \det_{X_1(p), \dots, X_n(p)} (v_1, \dots, v_n) \end{array} \right.$$

Then ω is nowhere vanishing and is thus a volume form. It follows that M is orientable.

2. Let $\pi_M \colon M \times N \to M$ and $\pi_N \colon M \times N \to N$ be the canonical projections. Let ω_M and ω_N be volume forms on M and N respectively. Let ω be defined by the relation $\omega = \pi_M^*(\omega_M) \wedge \pi_N^*(\omega_N)$. Let $(p,q) \in M \times N$, $\{v_1, \dots, v_m\}$ be a basis of T_pM, $\{w_1, \dots, w_n\}$ be a basis of T_qM. Then $\{(v_1, 0), \dots, (v_m, 0), (0, w_1), \dots, (0, w_n)\}$ is a basis of $T_{(p,q)}M \times N = T_pM \times T_qN$, and

$$\omega\big((v_1, 0), \dots, (v_m, 0), (0, w_1), \dots, (0, w_n)\big) = \omega_M(v_1, \dots, v_m) \times \omega_N(w_1, \dots, w_n) \neq 0$$

Hence, ω is a volume form on $M \times N$.

Remark. One can prove the result by exhibiting an oriented atlas. If $\{\varphi_i, U_i\}_{i \in I}$ and $\{V_j, \psi_j\}_{j \in J}$ are oriented atlas on M and N respectively, show that $\{(U_i \times V_j), (\varphi_i, \psi_j)\}_{(i,j) \in I \times J}$ is an oriented atlas on $M \times N$. To do so, show that

$$\det\left(\mathrm{d}\big((\varphi_{i_1}, \psi_{j_1}) \circ (\varphi_{i_2}, \psi_{j_2})^{-1}\big)\right) = \det\left(\mathrm{d}(\varphi_{i_1} \circ \varphi_{i_2}^{-1})\right) \det\left(\mathrm{d}(\psi_{j_1} \circ \psi_{j_2}^{-1})\right)$$

3. Recall that if $\{(U_i, \varphi_i)\}_{i \in I}$ is an atlas of M, then we can build an atlas $\{(TU_i, \Phi_i)\}_{i \in I}$ on TM by

$$\Phi_i \colon (p, v) \in TU_i \mapsto (\varphi_i(p), \mathrm{d}\varphi(p)v) \in \varphi_i(U) \times \mathbb{R}^n$$

whose transition functions are given by

$$\Phi_i \circ \Phi_j^{-1}(x, w) = \big(\varphi_i \circ \varphi_j^{-1}(x), \mathrm{d}(\varphi_i \circ \varphi_j^{-1})(x)w\big)$$

Then, by construction,

$$\mathrm{d}(\Phi_i \circ \Phi_j^{-1})(x, w)(w_1, w_2) = \left(\mathrm{d}(\varphi_i \circ \varphi_j^{-1})(x)w_1, \mathrm{d}(\varphi_i \circ \varphi_j^{-1})(x)w_2\right)$$

and finally,

$$\det\left(\mathrm{d}(\Phi_i \circ \Phi_j^{-1})(x, w)\right) = \det\left(\mathrm{d}(\varphi_i \circ \varphi_j^{-1})(x)\right)^2 > 0$$

Thus, TM is orientable.

Exercise 2.

Let $n \in \mathbb{N}$.

1. Show that the sphere $\mathbb{S}^n$ is orientable. Is the diffeomorphism $x \mapsto -x$ orientation preserving?

2. Is the projective space $\mathbb{R}P^n$ orientable?

Exercise 3.

Let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ and $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ on $\mathbb{R}^3$. Define α by

$$\forall Y, Z \in \Gamma(T\mathbb{R}^3), \quad \alpha(Y, Z) = \frac{1}{3}\omega(X, Y, Z)$$

1. Show that α is a differential form of degree 2. Give an expression of α in the basis $\{\mathrm{d}x \wedge \mathrm{d}y, \mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x\}$.
2. Does there exist β a differential form of degree 1 such that $\alpha = \mathrm{d}\beta$?
3. Let B be the open unit ball of $\mathbb{R}^3$. Compute $\int_B \mathrm{d}\alpha$.
4. Let $i\colon \mathbb{S}^2 \to \mathbb{R}^3$ be the inclusion of the unit sphere. Compute $\int_{\mathbb{S}^2} i^*\alpha$.

Exercise 4.

1. Let $f\colon \mathbb{R} \to \mathbb{R}_+^*$ be defined as $f(t) = e^t$. Compute $f^*(\frac{\mathrm{d}x}{x})$.
2. Let $f\colon (0, +\infty) \times (0, 2\pi) \to \mathbb{R}^2 \setminus \{0\}$ be defined as $f(r, \theta) = (r\cos\theta, r\sin\theta)$. Compute $f^*(\mathrm{d}x \wedge \mathrm{d}y)$.

Exercise 5.

Let $M = \mathbb{R}^2 \setminus \{0\}$. Consider the differential form of degree 1 $\alpha = -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$.

1. Compute $\mathrm{d}\alpha$.
2. Let C_0 be the circle centered at the origin of radius 1 and C_1 be the circle centered at $(3, 0)$ of radius 2. Let i_0 and i_1 be the respective inclusion maps. Compute $\int_{C_0} i_0^*\alpha$ and $\int_{C_1} i_1^*\alpha$.