A construction of the Möbius strip

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Consider the unidimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $p \colon \mathbb{R} \to \mathbb{T}$ the canonical projection, and the trivial rank 2 vector bundle $\mathbb{T} \times \mathbb{R}^2$. Consider the subset

$$E = \{(\bar{t}, \lambda(\cos \pi t, \sin \pi t)) \mid \bar{t} \in \mathbb{T}, \lambda \in \mathbb{R}\} \subset \mathbb{T} \times \mathbb{R}^2.$$

An element of E is a couple $(\bar{t}, v)$ where $\bar{t}$ lies in the torus $\mathbb{T}$ and v lies in the line of $\mathbb{R}^2$ making an angle of πt with the horizontal axis. One can easily check that E is a smooth manifold by finding local parametrizations (*hint*: $(t, \lambda) \mapsto \ldots$).

Let us show that E is a vector bundle on $\mathbb{T}$. Let $\pi \colon E \to \mathbb{T}$ be the projection onto the first factor. Then for $\bar{t} \in \mathbb{T}$, it holds that

$$E_{\bar{t}} = \pi^{-1}(\{\bar{t}\}) = \{\bar{t}\} \times \mathbb{R}(\cos \pi t, \sin \pi t) \simeq \mathbb{R}$$

and the fibers are vector spaces. Let us find trivializations of E. Consider $U = p((0,1))$ and $V = p\left((-\frac{1}{2}, \frac{1}{2})\right)$. Define

$$\begin{array}{rccc} \chi_U \colon & \pi^{-1}(U) & \longrightarrow & U \times \mathbb{R} \\ & (\bar{t}, v) & \longmapsto & (\{t\}, \langle v, (\cos \pi t', \sin \pi t') \rangle) \end{array}$$

where t' is the unique representant of $\bar{t}$ in $(0,1)$. Similarly, define

$$\begin{array}{rccc} \chi_V \colon & \pi^{-1}(V) & \longrightarrow & V \times \mathbb{R} \\ & (\bar{t}, v) & \longmapsto & (\{t\}, \langle v, (\cos \pi t'', \sin \pi t'') \rangle) \end{array}$$

where t'' is the unique representant of $\bar{t}$ in $(-\frac{1}{2}, \frac{1}{2})$. Indeed, the restriction of χ_U and χ_V on fibers $E_{\bar{t}}$ are linear isomorphisms, and if π_1 denotes the projection onto the first factor, then $\pi_1 \circ \chi_U = \mathrm{id}_U$, $\pi_1 \circ \chi_V = \mathrm{id}_V$.

Moreover, $U \cap V = p\left((0,1) \setminus \{\frac{1}{2}\}\right)$ and we have

$$\begin{array}{rccc} \chi_U \circ \chi_V^{-1} \colon & (U \cap V) \times \mathbb{R} & \longrightarrow & (U \cap V) \times \mathbb{R} \\ & (\bar{t}, v) & \longmapsto & \begin{cases} (\bar{t}, v) & \text{if } \bar{t} \in p\left((0, \frac{1}{2})\right) \\ (\bar{t}, -v) & \text{if } \bar{t} \in p\left((\frac{1}{2}, 1)\right) \end{cases} \end{array}$$

. Thus, E is a vector bundle over $\mathbb{T}$. Notice that the transition function is in fact given by

$$\begin{array}{rccc} g_{U,V} \colon & U \cap V & \longrightarrow & GL_1(\mathbb{R}) = \mathbb{R}^* \\ & \bar{t} & \longmapsto & \begin{cases} 1 & \text{if } \bar{t} \cap (0, \frac{1}{2}) \neq \varnothing \\ -1 & \text{if } \bar{t} \cap (\frac{1}{2}, 1) \neq \varnothing \end{cases} \end{array}$$

Remark: E can be seen by by different ways:

- as the quotient space of the action $\mathbb{Z} \curvearrowright \mathbb{R}^2$ given by $k \cdot (x, y) = (x + k, (-1)^k y)$,
- as the quotient of $[0,1] \times \mathbb{R}$ by the equivalence relation $(0, y) \sim (1, -y)$.